PARTICULAR SOLUTIONS OF SECOND ORDER ODES

Suppose we are asked to solve the inhomogeneous second-order differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \alpha\frac{\mathrm{d}}{\mathrm{d}x}y + \beta y = F(x),$$

where $F(x)$ is an elementary function. We know that given any particular solution $y_p(x)$ of the equation, the general solution will be given by

$$y(x) = y_p(x) + C_1u_1(x) + C_2u_2(x),$$

where C_1 and C_2 are arbitrary constants, and u_1, u_2 are solutions of the homogeneous equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \alpha\frac{\mathrm{d}}{\mathrm{d}x}y + \beta y = 0.$$

The solutions of the latter can be found easily in terms of the roots of the characteristic polynomial:

$$r^2 + \alpha r + \beta.$$

Here we indicate how to find a particular solution $y_p(x)$ for some simple examples of functions $F(x)$ (see Tables 1 and 2 below). Given the general form of $y_p(x)$, one uses the equation to determine the constants A, B, $A_n, A_{n-1}, \dots A_0$ and $B_n, B_{n-1}, \dots, B_0$ and determine $y_p(x)$ A distinction must be made between the case when $F(x)$ is a solution of the homogeneous equation and when it is not. In terms of the simple functions below, this condition can be expressed as r_0 not being a root of $r^2 + \alpha r + \beta$.. **In Table 1, we assume r_0 is *not* a root of the characteristic polynomial $r^2 + \alpha r + \beta$. In Table 2, we assume r_0 *is* a root the characteristic polynomial**. Note that for us, n in the tables below will typically be smaller than 2.

Let us briefly explain how one can arrive at the results in the tables with an example.[1] Consider the differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y - 6\frac{\mathrm{d}}{\mathrm{d}x}y + 9y = xe^x.$$

The characteristic polynomial

$$r^2 - 6x + 9 = (r-3)^2$$

has a double root $r = 3$. We know that this implies that the general solution of the homogeneous problem

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y - 6\frac{\mathrm{d}}{\mathrm{d}x}y + 9y = 0$$

has the form

$$C_1e^{3x} + C_2xe^{3x}.$$

[1]These explanations are only provided to help you understand how the form of y_p was found. You do not need to know this for the exam; in particular, we have not even discussed higher order equations in class.

To find a solution of the inhomogeneous problem, we notice that xe^x itself is a solution of a 2nd order differential equation. For clarity, let us denote $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then we have

$$(D^2 - 2D + 1)xe^x = (D-1)^2 xe^x = 0.$$

The idea is now to apply $(D-1)^2$ to both sides of our original equation:

$$\begin{aligned} D^2y - 6Dy + 9y &= xe^x \\ \Rightarrow (D-1)^2(D^2 - 6D + 9)y &= (D-1)^2 xe^x = 0. \end{aligned}$$

We now see that if y is to satisfy the inhomogeneous 2nd order equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y - 6\frac{\mathrm{d}}{\mathrm{d}x}y + 9y = xe^x,$$

it must also satisfy the 4th order homogeneous equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}y - 6\frac{\mathrm{d}}{\mathrm{d}x} + 9\right) y = 0.$$

We have not considered such equations in class, but one can also solve them using their characteristic polynomial. We can rewrite the equation above as

$$(D-1)^2(D-3)^2 y = 0,$$

which shows that its characteristic polynomial has two double roots, 1 and 3. In such a case, the general solution of the fourth order equation is

$$y(x) = C_1e^x + C_2xe^x + C_3e^{3x} + C_4xe^{3x}.$$

Note how this solution was found by the same method we use to solve 2nd order homogeneous equations:

(1) Identify the characteristic polynomial
(2) Each root r of the characteristic polynomial corresponds to a solution of the form
$$Ce^{rx}$$
for some constant C.
(3) Double roots give rise to an additional solution
$$Cxe^{rx}.$$

The above suggests that we look for a particular solution y_p of our equation which has the form

$$y_p(x) = C_1e^x + C_2xe^x,$$

which is what appears in the table. The constants C_1 and C_2 are now determined by the condition

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y_p - 6\frac{\mathrm{d}}{\mathrm{d}x}y_p + 9y_p = xe^x.$$

Suppose that, instead of xe^x on the right side, we had been asked to solve

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y - 6\frac{\mathrm{d}}{\mathrm{d}x}y + 9y = xe^{3x}.$$

In this case, xe^{3x} solves the homogeneous equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(xe^{3x}) - 6\frac{\mathrm{d}}{\mathrm{d}x}(xe^{3x}) + 9xe^3 = 0.$$

In other words, 3 is a root of the characteristic polynomial. The last equation can be rewritten as

$$(D-3)^2 xe^{3x} = 0.$$

Proceeding as in the previous paragraph, we find that the solution of the inhomogeneous 2nd order equation solves a homogeneous 4th order equation

$$(D-3)^2(D-3)^2 y = 0.$$

This equation has characteristic polynomial $(r-3)^4$; 3 has become a quadruple root. Inspired by what we did in the case of a double root, we might guess that the general solution has the form

$$y(x) = C_1 e^{3x} + C_2 xe^{3x} + C_3 x^2 e^{3x} + C_4 x^3 e^{3x}.$$

This guess turns out to be correct. This all suggests our particular solution will be of the form

$$y_p(x) = C_3 x^2 e^{3x} + C_4 x^3 e^{3x},$$

which agrees with the result in the table. Again, to identify our particular solution we need to solve for the coefficients C_3 and C_4 by plugging into the original equation.

TABLE 1. Case 1: r_0 is *not* a root of the characteristic polynomial.

$F(x)$	General form of $y_p(x)$
$e^{r_0 x}$	$Ae^{r_0 x}$
$x^n e^{r_0 x}$	$A_n x^n e^{r_0 x} + A_{n-1} x^{n-1} e^{r_0 x} + \ldots + A_0 e^{r_0 x}$
$\sin r_0 x$ **or** $\cos r_0 x$	$A \cos r_0 x + B \sin r_0 x$
$x^n \sin r_0 x$ **or** $x^n \cos r_0 x$	$A_n x^n \cos r_0 x + A_{n-1} x^{n-1} \cos r_0 x + \ldots + A_0 \cos r_0 x + B_n x^n \sin r_0 x + B_{n-1} x^{n-1} \sin r_0 x + \ldots + B_0 \sin r_0 x.$
$e^{ax} \sin bx$ **or** $e^{ax} \cos bx$, $(r_0 = a + bi)$	$Ae^{ax} \sin bx + Be^{ax} \cos(bx)$
$x^n e^{ax} \sin(bx)$ **or** $x^n e^{ax} \cos(bx)$, $(r_0 = a + bi)$	$A_n x^n e^{ax} \sin(bx) + A_{n-1} x^{n-1} e^{ax} \sin(bx) + \ldots + A_0 e^{ax} \sin(bx) + B_n x^n e^{ax} \cos(bx) + B_{n-1} x^{n-1} e^{ax} \cos(bx) + \ldots + B_0 e^{ax} \cos bx$
x^n, $(r_0 = 0)$	$A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0$

TABLE 2. Case 2: r_0 *is* a root of the characteristic polynomial.

$F(x)$	General form of $y_p(x)$
$e^{r_0 x}$	$Axe^{r_0 x}$
$x^n e^{r_0 x}$	$A_{n+1}x^{n+2}e^{r_0 x} + A_n x^{n+1}e^{r_0 x} + \ldots + A_0 x e^{r_0 x}$
$\sin r_0 x$ **or** $\cos(r_0 x)$	$Ax\cos r_0 x + Bx\sin r_0 x$
$x^n \sin r_0 x$ **or** $x^n \cos r_0 x$	$A_n x^{n+1}\cos r_0 x + A_{n-1}x^n \cos r_0 x + \ldots + A_0 x\cos r_0 x + B_n x^{n+1}\sin r_0 x + B_{n-1}x^{n-1}\sin r_0 x + \ldots + B_0 x \sin r_0 x.$
$e^{ax}\sin(bx)$ **or** $e^{bx}\cos(bx)$, $(r_0 = a + bi)$	$Axe^{ax}\sin(bx) + Bxe^{ax}\cos(bx)$
$x^n e^{ax}\sin(bx)$ **or** $x^n e^{ax}\cos(bx)$, $(r_0 = a + bi)$	$A_n x^{n+1}e^{ax}\sin(bx) + A_{n-1}x^{n-1}e^{ax}\sin(bx) + \ldots + A_0 x e^{ax}\sin(bx) + B_n x^{n+1}e^{ax}\cos(bx) + B_{n-1}x^{n-1}e^{ax}\cos(bx) + \ldots + B_0 x e^{ax}\cos bx$
x^n, $(r_0 = 0)$	$A_{n+1}x^{n+1} + A_{n-1}x^n + \ldots A_0 x.$